Mid-semester Examination
Fourier Analysis
I SEMESTER, 2014-15

Max. marks: 100 Time limit: 3 hours

1. Let $\Phi_n, n = 1, 2, ...$ be a sequence of functions on $[-\pi, \pi]$ with the following properties: $\sup\{\int_{-\pi}^{\pi} |\Phi_n(t)|\, dt : n = 1, 2, ...\} < \infty$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_n(t)dt \to 1$ as $n \to \infty$ and $\sup\{|\Phi_n(t)| : \pi \geq t \geq \delta\} \to 0$ as $n \to \infty$ for each $\delta \in (0, \pi)$. If $f \in L^1([-\pi, \pi])$ prove that $f * \Phi_n \to f$ in $L^1([-\pi, \pi])$. [20]

2. If $f \in L^2([-\pi, \pi])$ and $f_k = f * f * ... * f$ ($k - fold$ convolution of f) show that $\|f_k\|_2^{1/k} \to \sup\{\left|\hat{f}(n)\right| : n \geq 1\}$ as $k \to \infty$. Hint: write $\|f_k\|_2$ in terms of the Fourier coefficients of f_k. [15]

3. Prove or disprove: if f is a periodic function of bounded variation then the Fourier series of f converges to f uniformly. [10]

4. If f and g are absolutely continuous on $[a, b]$ show that fg is also absolutely continuous.Use this to show that $\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)f'(x)dx$. [15]

5. Prove that $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6}$ for $0 \leq x \leq 2\pi$ and show that the formula does not hold for any $x \in \mathbb{R}\backslash[0, 2\pi]$. [20]

6. Let $\{c_n\}$ be a sequence of positive numbers decreasing to 0. For what value of x does the series
$\sum_{n=1}^{\infty} c_n \sin(nx)$ converge? Justify your claim. [20]